Problem Sheet 1

Exercise 1.1.

For $a, b \in \mathbb{R}$, let $\alpha \in L^1([a,b];\mathbb{R})$, $\beta \in C([a,b];[0,\infty))$ and $u \in C([a,b];\mathbb{R})$ satisfy

$$u_t \leq \alpha_t + \int_a^t \beta_s u_s ds \tag{1}$$

for all $t \in [a,b]$. Prove the following versions of Grönwall's Inequality (Lemmas 1.2.3, 1.2.4):

1. For all $t \in [a,b]$,
$$u_t \leq \alpha_t + \int_a^t \alpha_s \beta_s e^{\int_s^t \beta_r dr} ds.$$

2. Under the additional assumption that α is constant,[1] then for all $t \in [a,b]$,
$$u_t \leq \alpha e^{\int_a^t \beta_r dr}.$$

Exercise 1.2.

Give an example of a stochastic process X whose law is constant in time, but is not stationary (in the sense of Definition 2.2.5).

Exercise 1.3.

Consider an ODE
$$\dot{x}_t = f(x_t)$$
in $\mathbb{R}^d$ with Euclidean norm $\|\cdot\|$, $d \geq 1$, where f is Lipschitz with Lipschitz constant K in $\mathbb{R}^d$. We introduce the flow map associated to the ODE, which is a function Φ defined for times s, t and a point $u \in \mathbb{R}^d$ as the solution x of the ODE at time t with 'initial condition' $x_s = u$. We denote this by $\Phi_{s,t}(u)$ and in the case where $s = 0$, simply $\Phi_t(u)$. Prove the following:

1. For any $s < t \in \mathbb{R}$, $\Phi_{s,t}(u) = \Phi_{t-s}(u)$;

2. For any $0 \leq s < t$, $\Phi_{s,t}(\Phi_s(u)) = \Phi_t(u)$;

3. For any $t \geq 0$,
$$\sup_{r\in[0,t]} \|\Phi_r(u) - \Phi_r(v)\| \leq \|x - y\| e^{Kt}.$$

Exercise 1.4.

Suppose we have a pair of real valued ODEs,

$$\dot{x}_t = f(x_t, z^1)$$
$$\dot{y}_t = f(y_t, z^2)$$

[1] In fact one only needs that α is non-decreasing, and the result holds for α_t replacing α.

where $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is K-Lipschitz in each variable, uniformly across the other variable: that is for every $a \in \mathbb{R}$ the functions $f(a, \cdot)$ and $f(\cdot, a)$ are both K-Lipschitz. Here $z^1, z^2 \in \mathbb{R}$ are given. Let Φ^x denote the flow for x and Φ^y denote the flow for y. Prove that for any $t \geq 0$,

$$\sup_{r \in [0,t]} |\Phi^x_r(u^1) - \Phi^y_r(u^2)| \leq \left(|u^1 - u^2| + K|z^1 - z^2|t\right) e^{Kt}.$$

Exercise 1.5.

Consider the real valued ODE

$$\dot{x}_t = -\alpha x_t + \beta \dot{f}_t$$

where $f \in C^1([0, \infty); \mathbb{R})$, $\alpha \in \mathbb{R}$ and initial condition $x_0 \in \mathbb{R}$. Show that for any $t \geq 0$ we have the identity

$$x_t = e^{-\alpha t} x_0 + \beta \int_0^t e^{-\alpha(t-s)} \dot{f}_s ds. \tag{2}$$

In particular,

$$x_t = e^{-\alpha t} x_0 + \beta \int_0^t e^{-\alpha(t-s)} df_s$$

where the last term is defined in the Riemann-Stieltjes sense.

Exercise 1.6. (Hard, involving stopping times which will be covered later in the course, but an interesting comparison to the classical Grönwall Lemma!)**

Fix $t > 0$ and suppose that $\boldsymbol{x}, \boldsymbol{\eta}$ are real-valued, non-negative stochastic processes. Assume, moreover, that there exists constants $c', \hat{c}$ (allowed to depend on t) such that for $\mathbb{P} - a.e.\ \omega$,

$$\int_0^t \boldsymbol{\eta}_s(\omega) ds \leq c' \tag{3}$$

and for all stopping times $0 \leq \theta_j < \theta_k \leq t$,

$$\mathbb{E}\left(\sup_{r \in [\theta_j, \theta_k]} \boldsymbol{x}_r\right) \leq \hat{c}\mathbb{E}\left(\left(\boldsymbol{x}_{\theta_j} + 1\right) + \int_{\theta_j}^{\theta_k} \boldsymbol{\eta}_s \boldsymbol{x}_s ds\right) < \infty.$$

Prove that there exists a constant C dependent only on $c', \hat{c}, t$ such that

$$\mathbb{E}\left(\sup_{r \in [0,t]} \boldsymbol{x}_r\right) \leq C\left(\mathbb{E}(\boldsymbol{x}_0) + 1\right).$$